Five Year Integrated M.Sc. Examination, 2023
Semester-VII
Subject: Integral Equations and Calculus of Variations
Course Code: MT-4-7-5

Time: 3 Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Attempt any four questions.

1. State and prove Replacement lemma. Convert the following integral equations into a differential equation with appropriate initial conditions
$y(x) = \lambda \int_0^1 k(x,t)y(t)dt$
where, $k(x,t) = \begin{cases} x(1-t) & x \leq t \leq 1 \\ t(1-x) & 0 \leq t \leq x. \end{cases}$ 2+3+5=10

2. (i) Form the integral equation corresponding to
$y'' + \lambda y = 0, y(0) = 0, y(1) = 0.$

(ii) Find the eigen values and eigen functions of the integral equation

$$y(x) = \lambda \int_0^{2\pi} \cos(x+t)y(t)dt.$$

5+5=10

3. (i) Solve $y(x) = 2x - \pi + 4\int_0^{\frac{\pi}{2}} \sin^2 xy(t)dt$.
(ii) Show that the equation $y(x) = e^x + \lambda \int_0^1 (5x^2-3)t^2 y(t)dt$ has the unique solution.

5+5=10

4. Find the Resolvent kernel of the integral equation $y(x) = f(x) + \lambda \int_0^x y(t)dt$. Hence find the solution.

7+3=10

5. Show that $J = \int_0^{\frac{\pi}{2}} (y'^2 - y^2)dx$ has
(i) unique extremal at $y(0) = 0, y(\frac{\pi}{2}) = 1$
(ii) no extremal at $y(0) = 0, y(\pi) = 1$
(iii) infinite number of extremals at $y(0) = 0, y(\pi) = 0$.

4+3+3=10

6. (i) Find the extremals of the functional $\int_0^{\frac{\pi}{2}}(y'^2 + z'^2 + 2yz)dx$
where $y(0) = 0, y(\frac{\pi}{2}) = 1, z(0) = 0$, and $z(\frac{\pi}{2}) = -1$.
(ii) Solve the following variational problem by finding extremals satisfying the given conditions

$$I(y) = \int_0^1 (1 + (y'')^2)dx \qquad y(0) = 0, y'(0) = 1, y(1) = 1, y'(1) = 1.$$

5+5=10